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ON THE STUDY OF RANDOM OSCILLATIONS IN NON-AUTONOMOUS MECHANICAL SYSTEMS USING THE FOKKER-PLANCK-KOLMOGOROV EQUATIONS*

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A method of integrating the Fokker-Plank-Kolmogorov equations (FPKE) used in the theory of random oscillations /1—4/ is proposed. The Duffing equation is first studied as an example. The method is then used, together with the method of averaging, to study random oscillations of non-autonomous mechanical systems with one degree of freedom when the eigenfrequency varies in a random manner. The Van-der-Pol equation is considered for the case of a randomly varying eigenfrequency and periodic parametric excitation. When the function sought is replaced, the FPKE transform into another equation whose trivial solutions have the corresponding particular solutions of the FPKE. The condition of integrability of the FPKE is obtained as the direct consequence of the change in question.

1. Consider a mechanical system with one degree of freedom, whose motion is described by the following stochastic equation:

$$x'' + \omega^2 x = f(x, x') + \sigma\xi'(t) \tag{1.1}$$

$$f(x, x') = \sum_{s=1}^{m} \alpha_s \sum_{i,j=0}^{i+j=s} \gamma_{ij} x^i x'^j; \quad \alpha_s, \gamma_{ij} = \text{const} \tag{1.2}$$

where $\xi'(t)$ is a random, white noise-type action of unit intensity. Using the substitution

$$x = a\cos\psi, \quad x' = -a\omega\sin\psi \tag{1.3}$$

and the Ito formula, we reduce Eq.(1.1) to the form /4/

$$da = \left[-\frac{1}{\omega} f(a\cos\psi, -a\omega\sin\psi)\sin\psi + \frac{\sigma^2}{2a\omega^2}\cos^2\psi\right]dt - \frac{\sigma}{\omega}\sin\psi\, d\xi(t) \tag{1.4}$$

$$d\psi = \left[\omega - \frac{1}{a\omega} f(a\cos\psi, -a\omega\sin\psi)\cos\psi - \frac{\sigma^2}{a^2\omega^2}\sin\psi\cos\psi\right]dt - \frac{\sigma}{a\omega}\cos\psi\, d\xi(t)$$

Let us write the FPKE corresponding to system (1.4) for the stationary probability density of the amplitude and phase $W(a, \psi)$

$$\frac{\partial}{\partial a}[B_1(a,\psi)W] + \frac{\partial}{\partial\psi}[B_2(a,\psi)W] = \frac{1}{2}\left\{\frac{\partial^2}{\partial a^2}[B_{11}(a,\psi)W] + 2\frac{\partial^2}{\partial a\,\partial\psi}[B_{12}(a,\psi)W] + \frac{\partial^2}{\partial\psi^2}[B_{22}(a,\psi)W]\right\} \tag{1.5}$$

Taking into account the expression for $f(x, x')$ (1.2), we obtain

$$B_1(a,\psi) = -\frac{\sin\psi}{\omega} f(a\cos\psi, -a\omega\sin\varphi) + \frac{\sigma^2\cos^2\psi}{2\omega^2 a} = \frac{\sigma^2\cos^2\psi}{2\omega^2}a^{-1} + \sum_{s=1}^{m} A_s(\psi)a^s \tag{1.6}$$

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$$A_s(\psi) = \alpha_s \sum_{i,\, j=0}^{i+j=s} \gamma_{ij} (-\omega)^{j-1} \cos^i \psi \sin^{j+1}\psi$$

$$B_1(a, \psi) = \omega - \frac{\sigma^2 \cos\psi \sin\psi}{\omega^2 a^2} - \frac{\cos\psi}{a\omega} f(a\cos\psi, -a\omega\sin\psi) =$$

$$\omega - \frac{\sigma^2 \cos\psi \sin\psi}{\omega^2} a^{-2} + \sum_{s=1}^{m} D_s(\psi)\, a^{s-1}$$

$$D_s(\psi) = \alpha_s \sum_{i,\, j=0}^{i+j=s} \gamma_{ij} (-\omega)^{j-1} \cos^{i+1}\psi \sin^j\psi$$

$$B_{11}(a, \psi) = \frac{\sigma^2}{\omega^2} \sin^2\psi, \quad B_{12}(a, \psi) = \frac{\sigma^2 \sin\psi \cos\psi}{a\omega^2}$$

$$B_{22}(a, \psi) = \frac{\sigma^2 \cos^2\psi}{a^2\omega^2}$$

Let us make the substitution

$$W(a, \psi) = \exp\{\Phi(a, \psi)\} \tag{1.7}$$

Taking into account the identities

$$\frac{1}{W}\frac{\partial W}{\partial a} = \frac{\partial \Phi}{\partial a}, \quad \frac{1}{W}\frac{\partial^2 W}{\partial a\, \partial\psi} = \frac{\partial \Phi}{\partial a}\frac{\partial \Phi}{\partial \psi} + \frac{\partial^2 \Phi}{\partial a\, \partial \psi}$$

we transform Eq.(1.5) to the form

$$\left(\frac{\partial B_1}{\partial a} + \frac{\partial B_2}{\partial \psi} - \frac{\partial^2 B_{12}}{\partial\psi\, \partial a} - \frac{1}{2}\frac{\partial^2 B_{22}}{\partial \psi^2}\right) + \left(B_1 - \frac{\partial B_{12}}{\partial \psi}\right)\frac{\partial\Phi}{\partial a} + \tag{1.8}$$

$$\left(B_2 - \frac{\partial B_{12}}{\partial a} - \frac{\partial B_{22}}{\partial \psi}\right)\frac{\partial \Phi}{\partial \psi} - \frac{B_{11}}{2}\left[\left(\frac{\partial\Phi}{\partial a}\right)^2 + \frac{\partial^2\Phi}{\partial a^2}\right] -$$

$$B_{12}\left[\left(\frac{\partial\Phi}{\partial a}\right)\left(\frac{\partial\Phi}{\partial\psi}\right) + \frac{\partial^2\Phi}{\partial a\, \partial\psi}\right] - \frac{B_{22}}{2}\left[\left(\frac{\partial\Phi}{\partial\psi}\right)^2 + \frac{\partial^2\Phi}{\partial\psi^2}\right] = 0$$

The problem consists of solving the non-linear partial differential equation (1.8) whose coefficients are found from (1.6). First we note that the amplitude a, in Eq.(1.8) with the coefficients (1.6), can play the part of the generalized ignorable coordinate /5—7/ (the coefficients of the equation are polynomials with integral powers of the amplitude a). Consequently, using the method of expansion in series in terms of the generalized ignorable coordinate /5—7/, we can also find the solution of (1.8) in $\partial\Phi/\partial a$ in the form of a polynomial containing integral powers of the amplitude a. Then we have

$$\Phi(a, \psi) = \ln C + \ln a + \sum_{i=1}^{\infty} \mu_i(\psi)\, a^i, \quad C = \text{const}, \quad C > 0 \tag{1.9}$$

Substituting the expressions for Φ from (1.9) and for the coefficients $B_1, B_2, B_{11}, B_{12}, B_{22}$ from (1.6) into (1.8), and equating the coefficients of $a^{-2}, a^{-1}, a^0, a^1, \ldots$, we obtain the following set of equations for the unknown $\mu_1, \mu_2, \mu_3, \ldots$ (a prime denotes differentiation with respect to ψ)

$$\frac{\sigma^2}{2\omega^2}(\cos^2\psi - 2\sin^2\psi) + \frac{\sigma^2}{2\omega^2}(2\sin^2\psi - \cos^2\psi) = 0 \tag{1.10}$$

$$\frac{\sigma^2\cos^2\psi}{2\omega^2}(\mu_1 + \mu_1'') = 0$$

$$\frac{\sigma^2}{\omega^2}\left(\frac{\cos^2\psi}{2}\mu_2'' + \sin\psi\cos\psi\,\mu_2' + \mu_2\right) = 2A_1(\psi) + D_1'(\psi) -$$

$$\frac{\sigma^2}{\omega^2}\left[\frac{\sin^2\psi}{2}\mu_1^2 + \frac{\cos^2\psi}{2}\mu_1'^2 + \sin\psi\cos\psi\,\mu_1\mu_1'\right]$$

$$\frac{\sigma^2}{\omega^2}\left(\frac{\cos^2\psi}{2}\mu_3'' + 2\sin\psi\cos\psi\,\mu_3' + \frac{3(1+\sin^2\psi)}{2}\mu_3\right) =$$

$$3A_2(\psi) + D_2'(\psi) + A_1(\psi)\mu_1 - \frac{\sigma^2}{\omega^2}[2\sin^2\psi\,\mu_1\mu_2 - \cos^2\psi\,\mu_1'\mu_2 +$$

$$2\sin\psi\cos\psi\,\mu_1'\mu_2 + \sin\psi\cos\psi\,\mu_2'\mu_1] + (\omega + D_1(\psi))\mu_1'$$

$$\frac{\sigma^2}{\omega^2}\left(\frac{\cos^2\psi}{2}\mu_{n+2}'' + (n-1)\sin\psi\cos\psi\,\mu_{n+2}' +\right.$$

$$\left.\frac{n+2}{2}(1+n\sin^2\psi)\mu_{n+2}\right) = (n+2)A_{n+1}(\psi) +$$

$$D_{n+1}'(\psi) - \sum_{i,\, s=1}^{i+s=n+1}[iA_s(\psi)\mu_i + D_s(\psi)\mu_i'] +$$

$$\omega\mu_n' - \frac{\sigma^2}{\omega^2}\left[\frac{\sin^2\psi}{2}\sum_{i,\, j=1}^{i+j=n+2} ij\mu_i\mu_j - \sin\psi\cos\psi\sum_{i,\, j=1}^{i+j=n+2} i\mu_i\mu_j' + \frac{\cos^2\psi}{2}\sum_{i,\, j=1}^{i+j=n+2}\mu_i'\mu_j'\right], \quad n = 2, 3, \ldots$$

$$A_k(\psi),\ D_k(\psi) = 0, \quad k = m+1,\ m+2, \ldots$$

The differential equations of this system are separable in μ_i and enable all the $\mu_1, \mu_2, \mu_3, \ldots$ to be found in succession. The arbitrary integration constants must be chosen from the condition for the functions $\mu_i(\psi)$ to be the periodic. The problem of the convergence of the series (1.9) is not easy, and system (1.10) must, in general, be considered separately.

We shall merely note that the problem of convergence does not arise in cases when system (1.10) admits of an exant solution of the form

$$\mu_i = \varphi_i(\psi), \quad i = 1, 2, \ldots, N; \quad \mu_l = 0, \quad l \geqslant N+1$$

Then we obtain

$$W(a, \psi) = Ca \exp\left\{\sum_{i=1}^{N} \varphi_i(\psi)\, a^i\right\}$$

This property will be shared by many mechanical systems with one degree of freedom whose non-linear part depends linearly on the velocity.

We shall consider, as an example, Eq.(1.1) when

$$f(x, x^{\cdot}) = -\beta x^{\cdot} - \gamma x^3;\ \beta, \gamma > 0 \tag{1.11}$$

(Duffing's equation). In this case (1.6) yields

$$\begin{aligned} &A_1 = -\beta \sin^2\psi, \quad A_2 = 0, \quad A_3 = \gamma\omega^{-1}\cos^3\psi \sin\psi \\ &D_1 = -\beta \sin\psi\cos\psi, \quad D_2 = 0, \quad D_3 = \gamma\omega^{-1}\cos^4\psi \\ &A_n = 0, \quad D_n = 0, \quad n \geqslant 4 \end{aligned} \tag{1.12}$$

Substituting expressions (1.12) into (1.10), we can show that it admits of the following solution:

$$\begin{aligned} &\mu_1 = \mu_3 = 0, \quad \mu_2 = -\beta\omega^2\sigma^{-2} \\ &\mu_4 = -{}^1/_2\beta\gamma\sigma^{-2}\cos^4\psi; \quad \mu_s = 0, \quad s \geqslant 5 \end{aligned}$$

Therefore the solution of (1.5) corresponding to the Duffing case (1.11) will be

$$W(a, \psi) = Ca \exp\left\{-\frac{\beta\omega^2}{\sigma^2}a^2 - \frac{\beta\gamma\cos^4\psi}{2\sigma^2}a^4\right\} \tag{1.13}$$

We note that solution (1.13) can also be obtained from the known solution of the FPKE corresponding to the Duffing case (1.1) by changing from the phase coordinates $x, x^{\cdot}$ to amplitude and phase coordinates /2/.

2. Let us now use the method of solving the FPKE given in Sect.1, together with the principle of averaging, to study random oscillations of non-autonomous mechanical systems with one degree of freedom, with a randomly varying frequency /3/, described by the following stochastic differential equation:

$$x^{\cdot\cdot} + [\omega^2 + \sqrt{\varepsilon}\sigma\xi^{\cdot}(t)]x = \varepsilon f(x, x^{\cdot}, \omega t) \tag{2.1}$$

$$f(x, x^{\cdot}, \omega t) = \sum_{s=1}^{m} \alpha_s \sum_{i+j=0}^{i+j=s} \gamma_{ij} x^i x^{\cdot j} + \gamma x^2 \cos\omega t \tag{2.2}$$

where $\xi^{\cdot}(t)$ is a random, white noise-type action of unit intensity. Using the substitution (1.3) where $\psi = \omega t + \theta(t)$ and Ito's formula, we reduce Eq.(2.1) to the standard form

$$\begin{aligned} da &= \left[-\frac{\varepsilon}{\omega} f \sin\psi + \frac{\varepsilon\sigma^2\cos^4\psi}{2\omega^2}\right]dt + \frac{\sqrt{\varepsilon}\sigma}{\omega} a\cos\psi\sin\psi\, d\xi(t) \\ d\theta &= \left[-\frac{\varepsilon}{a\omega} f\cos\psi - \frac{\varepsilon\sigma^2}{\omega^2}\cos^2\psi\sin\psi\right]dt - \frac{\sqrt{\varepsilon}\sigma}{\omega}\cos^2\psi\, d\xi(t) \end{aligned} \tag{2.3}$$

Let us write the averaged FPKE corresponding to (2.3) for the stationary density of the amplitude and phase probabilities /4, 8/

$$\frac{\partial}{\partial a}(K_1 W) - \frac{\partial}{\partial\theta}(K_2 W) = \frac{1}{2}\frac{\partial^2}{\partial a^2}(K_{11}W) + \frac{\partial^2}{\partial a\,\partial\theta}(K_{12}W) + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}(K_{22}W) \tag{2.4}$$

Taking into account relation (2.2), we obtain /4/ expressions for $K_1, K_2, K_{11}, K_{12}, K_{22}$. Now, carrying out the substitution (1.7) we obtain, as in Sect.1, the equation for $\partial\Phi/\partial a$. We seek the solution of this equation in the form of a polynomial with integral powers of the amplitude a, i.e.

$$\Phi(a, \theta) = \lambda_1 \ln a + \sum_{i=0}^{\infty} \mu_i(\theta)\, a^i + \ln C, \quad C > 0, \quad \lambda_1 = \mathrm{const} \tag{2.5}$$

where $\mu_i(\theta)$ are unknown coefficients depending on the phase θ only. Finally, as in Sect.1, we obtain the following equations for $\lambda_1, \mu_0, \mu_1, \ldots$:

$$3\mu_0'' - \frac{16\omega\beta_1}{\sigma^2}\mu_0' + 3\mu_0'^2 + \lambda_1^2 + \frac{16\omega\eta_{11}}{\sigma^2}\lambda_1 + \frac{16\omega\eta_{11}}{\sigma^2} - 1 = 0 \tag{2.6}$$

$$3\mu_1'' + \left(6\mu_0' + \frac{16\omega\beta_1}{\sigma^2}\right)\mu_1' + \left(2\lambda_1 + \frac{16\omega\eta_1}{\sigma^2} + 1\right)\mu_1 =$$
$$\frac{16\omega^2}{\sigma^2}\left\{-\frac{2\eta_2}{\omega} + \frac{\gamma\sin\theta}{8\omega} - \left(\frac{\eta_2}{\omega} + \frac{\gamma\sin\theta}{8\omega}\right)\lambda_1 + \left(\frac{\beta_2}{\omega} + \frac{3\gamma\cos\theta}{8\omega}\right)\mu_0'\right\}$$
$$3\mu_k'' + \left(6\mu_0' + \frac{16\omega\beta_1}{\sigma^2}\right)\mu_k' + (k(k + 2\lambda_1 - 1)) + \left(\frac{16\omega\eta_1 k}{\sigma^2} + k\right)\mu_k =$$
$$\frac{16\omega^2}{\sigma^2}\left\{-\frac{(k+1)\eta_{k+1}}{\omega} + \left(\frac{\eta_2}{\omega} + \frac{\gamma\sin\theta}{8\omega}\right)(k-1)\mu_{k-1} -\right.$$
$$\sum_{i=0,\ s=3}^{s+i=k+1} \frac{i\mu_i\eta_s}{\omega} - \frac{\eta_{k+1}\lambda_1}{\omega} - \left(\frac{\beta_2}{\omega} + \frac{3\gamma\cos\theta}{8\omega}\right)\mu_{k-1}' -$$
$$\left.\sum_{i=0,\ s=3}^{s+i=k+1} \frac{\beta_s\mu_i'}{\omega}\right\} - \sum_{i=1,\ j=1}^{i+j=k} ij\mu_i\mu_j - 3\sum_{i=1,\ j=1}^{i+j=k} \mu_i'\mu_j', \quad k = 2, 3, \ldots$$

Thus we have obtained a closed system of differential equations which enable us to determine all λ_1, μ_i in succession. The arbitrary integration constants must be chosen from the condition for the function $\mu_i(\theta)$ to be periodic. The remarks concerning system (1.10) made in Sect.1 also apply to system (2.6).

System (2.6) admits, in many cases, of an exact solution of the form

$$\lambda_1 = \alpha = \text{const},\ \mu_i = \varphi_i(\theta),\ i = 0, 1, 2, \ldots, N;\ \mu_j = 0,\ j \geqslant N+1$$

Then we obtain the following exact solution of the FPKE (2.4):

$$W(a, \theta) = Ca^\alpha \exp\left\{\sum_{i=0}^{N} \varphi_i(\theta) a^i\right\}, \quad C > 0 \tag{2.7}$$

We shall consider, as an example, a non-autonomous Van-der-Pol equation for the case when the eigenfrequency varies randomly

$$x'' + (\omega^2 + \sqrt{\varepsilon}\sigma\xi^\cdot(t))x = \varepsilon\{\Delta x + (1 - \beta x^2)x' + \gamma x^2\cos\omega t\},\quad \omega^2 - \varepsilon\Delta = \nu^2 \tag{2.8}$$

The computations yield

$$K_1(a, \theta) = \frac{a}{2} - \frac{\beta a^3}{8} - \frac{\gamma a^2\sin\theta}{8\omega} + \frac{3\sigma^2}{16\omega^2}a$$
$$K_2(a, \theta) = \frac{\Delta}{2\omega} - \frac{3\gamma a}{8\omega}\cos\theta;\ \eta_1 = -\frac{\omega}{2},\ \eta_2 = 0,\ \eta_3 = \frac{\beta\omega}{8}$$
$$\eta_i = 0,\ i = 4, 5, \ldots,\ \beta_1 = -\frac{\Delta}{2},\ \beta_j = 0,\ j = 2, 3, \ldots$$

In the present case system (2.6) has an exact solution

$$\lambda_1 = \frac{8\omega^2}{\sigma^2} + 1,\ \mu_0 = \frac{8\Delta\omega}{3\sigma^2}\theta,\ \mu_1 = -\frac{2\omega\gamma}{\sigma^2}\sin\theta$$
$$\mu_2 = -\frac{\beta\omega^2}{\sigma^2},\ \mu_i = 0,\ i = 3, 4, \ldots \tag{2.9}$$

However, for the function $\mu_0(\theta)$ to be periodic, we must put $\Delta = 0$. Substituting (2.9) into (2.7) we obtain the exact solution of the FPKE corresponding to Eq.(2.9), with the exact principal resonance

$$W(a, \theta) = a^\varkappa \exp\left\{-\frac{2\omega\gamma\sin\theta}{\sigma^2}a - \frac{\beta\omega^2}{\sigma^2}a^2\right\}$$
$$\varkappa = 8\omega^2\sigma^{-2} + 1 \tag{2.10}$$

The probability density $W(a, \theta)$ attains its extremal value at the points where $\partial W/\partial a = \partial W/\partial\theta = 0$.

Stability of the stationary random oscillations can be attained under the condition that W reaches its maximum. We note that in the present case $W(0, \theta) = 0$, therefore the equilibrium position $x = 0$ is unstable.

In general, the inequality /3/ $\lambda_1 > 0$ represents the sufficient condition of instability of the equilibrium position.

3. Let us consider a non-automomous mechanical system with one degree of freedom in the principal region of resonance

$$x'' + \omega^2 x = \varepsilon[f(x, x', \omega t) - \Delta x] + \sqrt{\varepsilon}\sigma\xi^\cdot(t),\qquad \nu^2 = \omega^2 + \varepsilon\Delta \tag{3.1}$$

As in Sect.2, the substitution (1.3) where $\psi = \omega t + \theta$ transforms Eq.(3.1) to its standard form and the corresponding averaged FPKE, for the stationary probability density $W(a, \theta)$, take the form /4/

$$\frac{\partial}{\partial a}(K_1 W) + \frac{\partial}{\partial\theta}(K_2 W) - \frac{\sigma^2}{4\omega^2}\left[\frac{\partial^2 W}{\partial a^2} + \frac{1}{a^2}\frac{\partial^2 W}{\partial\theta^2}\right] = 0 \tag{3.2}$$

where

$$K_1(a,\theta) = M_t\left(\frac{\sigma^2}{2a\omega^2}\cos^2\psi - \frac{\sin\psi}{\omega}(f - \Delta a\cos\psi)\right) = \tag{3.3}$$
$$\frac{\sigma^2}{4a\omega^2} - \frac{1}{\omega}M_t(f\sin\psi)$$
$$K_2(a,\theta) = M_t\left(-\frac{\sigma^2\sin\psi\cos\psi}{a^2\psi^2} - \frac{\cos\psi}{a\omega}(f - \Delta a\cos\psi)\right) =$$
$$\frac{\Delta}{2\omega} - \frac{1}{a\omega}M_t(f\cos\psi)$$

In (3.2) let us make the substitution of the function in question

$$W(a,\theta) = S(a,\theta)\exp\left\{\frac{4\omega^2}{\sigma^2}\int_1^a K_1(a,\theta)\,da\right\} \tag{3.4}$$

Substituting expression (3.4) into (3.2) and carrying out simple transformations, we arrive at the following equation for the unknown function $S(a,\theta)$:

$$\left[\frac{\partial}{\partial\theta}\left(K_2 - \frac{1}{a^2}\int_1^a \frac{\partial K}{\partial\theta}\,da\right) + \frac{4\omega^2}{\sigma^2}\int_1^a \frac{\partial K_1}{\partial\theta}\,da\left(K_2 - \frac{1}{a^2}\int_1^a \frac{\partial K_1}{\partial\theta}\,da\right)\right]S - \tag{3.5}$$
$$K_1\frac{\partial S}{\partial a} + \left(K_2 - \frac{2}{a^2}\int_1^a \frac{\partial K_1}{\partial\theta}\,da\right)\frac{\partial S}{\partial\theta} - \frac{\sigma^2}{4\omega^2}\left(\frac{\partial^2 S}{\partial a^2} + \frac{1}{a^2}\frac{\partial^2 S}{\partial\theta^2}\right) = 0$$

We see that this equation admits of the trivial solution

$$S(a,\theta) = C,\quad C = \text{const},\quad C > 0 \tag{3.6}$$

provided that

$$\frac{\partial}{\partial\theta}\left(K_2 - \frac{1}{a^2}\int_1^a \frac{\partial K_1}{\partial\theta}\,da\right)\ \frac{4\omega^2}{\sigma^2} - \int_1^a \frac{\partial K_1}{\partial\theta}\,da\left(K_2 - \frac{1}{a^2}\int_1^a \frac{\partial K_1}{\partial\theta}\,da\right) \tag{3.7}$$

or, in particular, if

$$\frac{\partial}{\partial a}(a^2 K_2) = \frac{\partial K_1}{\partial\theta} \tag{3.8}$$

Substituting here the epxressions for K_1, K_2 from (3.3), we obtain the condition of integrability of the FPKE

$$\frac{\Delta a}{\omega} - \frac{1}{\omega}\frac{\partial}{\partial a}\{aM_t(f\cos\psi)\} = -\frac{1}{\omega}\frac{\partial}{\partial\theta}\{M_t(f\sin\psi)\} \tag{3.9}$$

When this condition holds, we obtain from (3.4), (3.6) the following solution for the corresponding FPKE:

$$W(a,\theta) = Ca\exp\left\{-\frac{4\omega}{\sigma^2}\int_1^a M_t\left[f(a\cos\psi, -a\omega\sin\psi, \psi - \theta)\sin\psi\right]da\right\} \tag{3.10}$$

It can be shown that the condition of integrability of (3.9) will be satisfied if /9/

$$\Delta = 0,\quad \alpha > 0 \tag{3.11}$$
$$f(x, x', \omega t) = \sum_{i=0}^{m}\left[g_i(x)x'^{2i+1} + h_i(x')x^{2i}\right] - \alpha\,\mathrm{sign}\,x' + P\cos\omega t$$

where $g_i(x), h_i(x')$ are polynomials in x and x' respectively. Substituting (3.11) into (3.10), we obtain the solution of the corresponding FPKE (3.2).

Let us consider, as an example, the stochastic equation

$$x'' + 2\varepsilon\alpha x' + \omega^2(1 + \varepsilon\lambda\cos 2\omega t)x = \varepsilon P\cos\nu t + \sqrt{\varepsilon}\sigma\xi'(t),\quad \alpha > 0 \tag{3.12}$$

Computations yield

$$K_1(a,\theta) = \frac{\sigma^2}{4a\omega^2} - \frac{P}{2\omega}\sin\theta - \alpha\omega + \frac{\omega\lambda a}{4}\sin 2\theta$$
$$K_2(a,\theta) = -\frac{P}{2a\omega}\cos\theta + \frac{\omega\lambda}{4}\cos 2\theta$$

It can be shown that in this case the condition of integrability of (3.8) holds. From the solution of FPKE corresponding to the equation (3.12)

$$W(a,\theta) = Ca\exp\left\{-\frac{2\omega P}{\sigma^2}a\sin\theta - \frac{\omega^2}{2\sigma^2}(4\alpha - \omega\lambda\sin 2\theta)a^2\right\}$$

it follows that the sufficient condition for a steady random oscillation to exist is that

$$4\alpha > \omega|\lambda|$$

The above inequality serves as the condtion of stability of system (3.12) when there are no external forces /1/.

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